Most flows are unstable...

Vortex shedding

Saffman-Taylor

Flow separation

Tollmien-Schlichting

Rayleigh-Taylor

Lift-up and Streaks

Traffic waves

Meandering instability

Gravito-capillary waves

Taylor-Couette

Rayleigh-Plateau

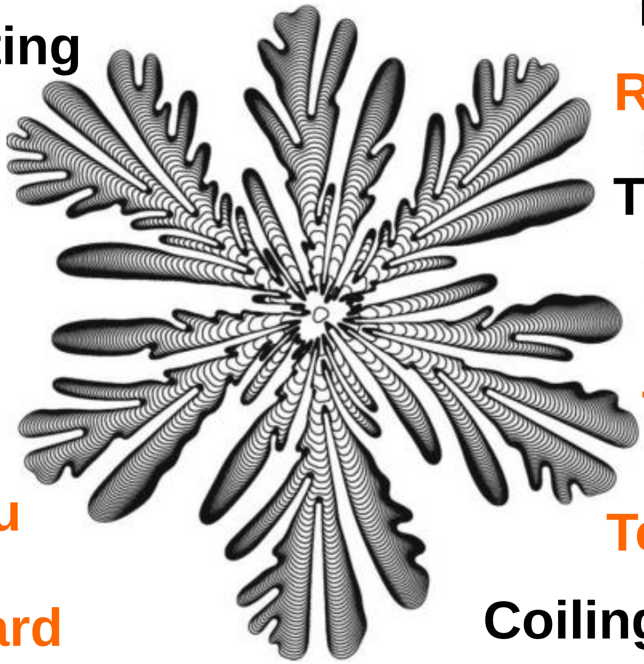
Tearing instability

Rayleigh-Benard

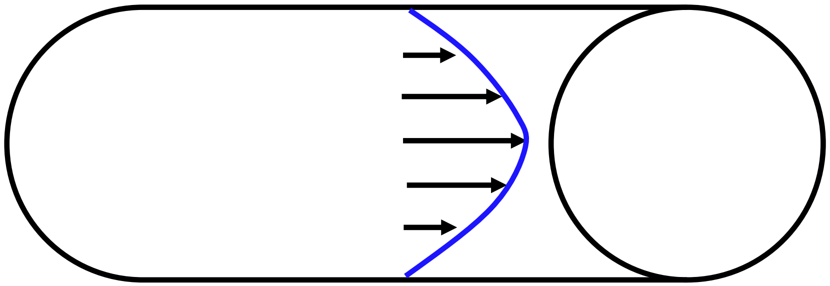
Coiling instability

Benard-Marangoni

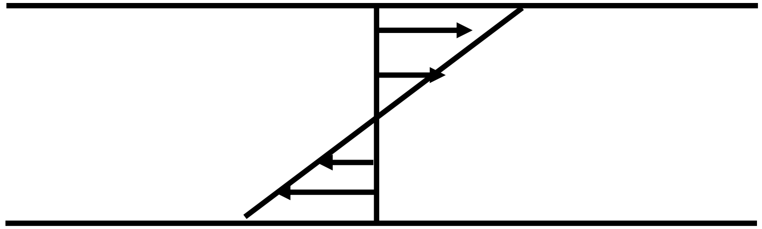
Kelvin-Helmholtz



Hagen Poiseuille



Couette



Flow type	Re_{exp}	Re_{lin}
Pipe flow	≈ 2000	∞
Plane Poiseuille flow	≈ 1000	5772
Plane Couette flow	≈ 360	∞

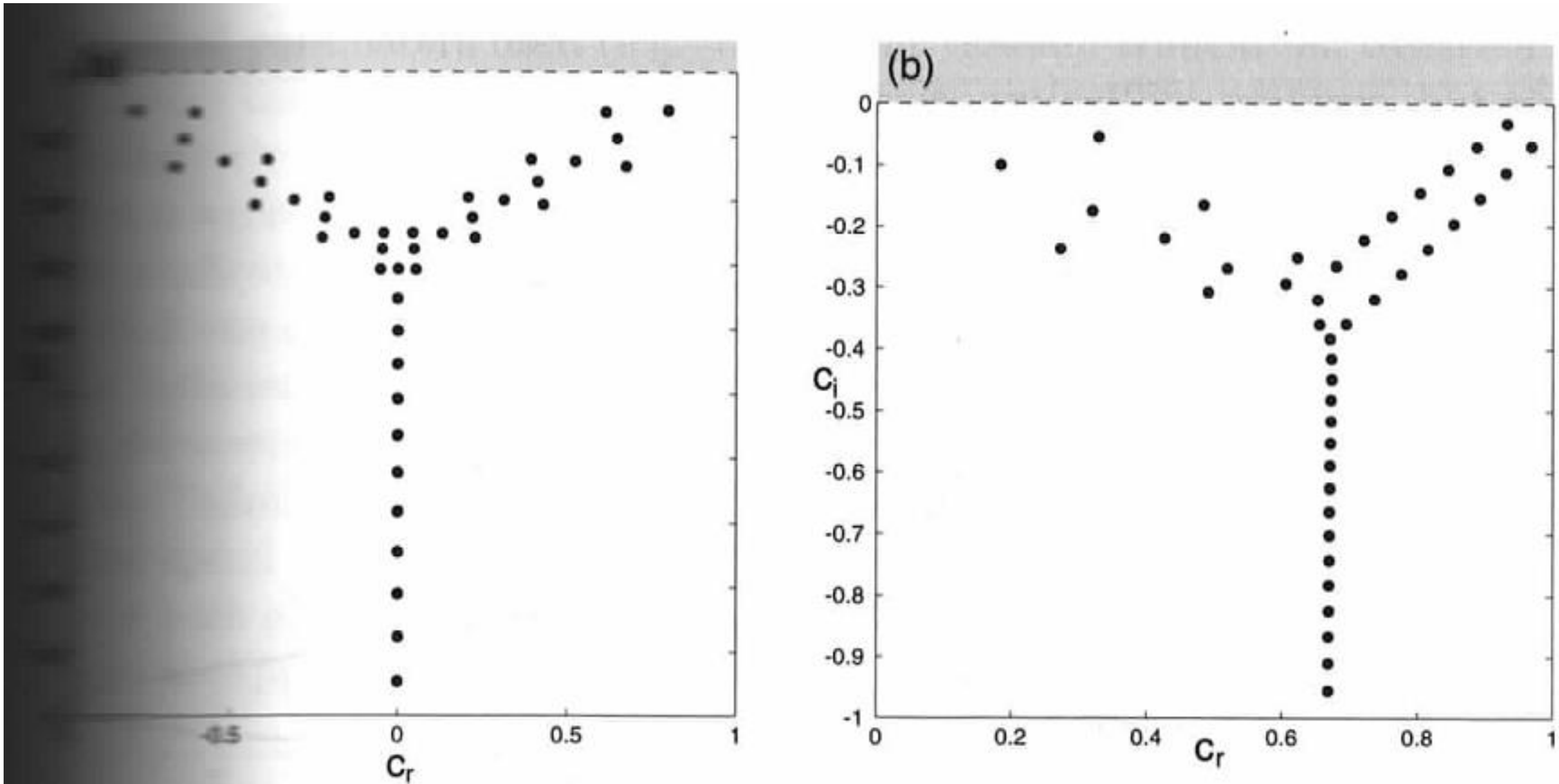


FIGURE 3.3. Spectrum of plane Couette and pipe flow. (a) Plane Couette flow for $\alpha = 1, \beta = 1, \mathrm{Re} = 1000$; (b) Pipe flow for $\alpha = 1, n = 1, \mathrm{Re} = 5000$.

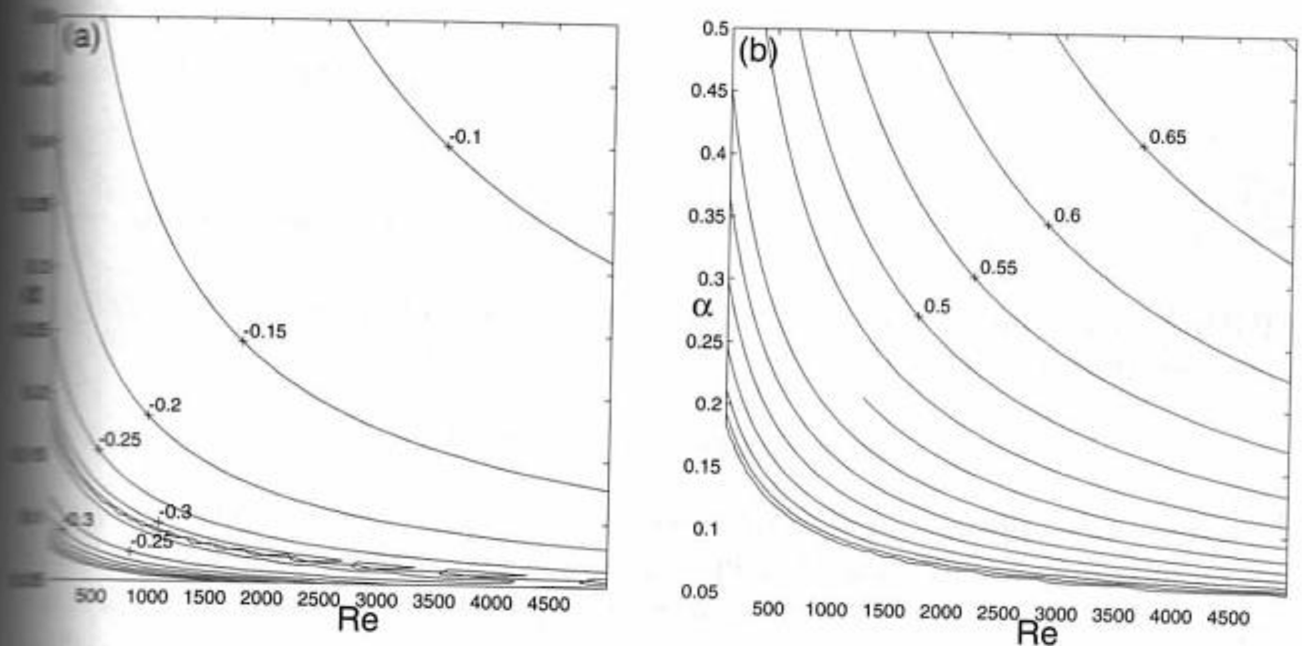


FIGURE 3.11. Contours of constant growth rate (a) and phase speed (b) for plane Couette flow.

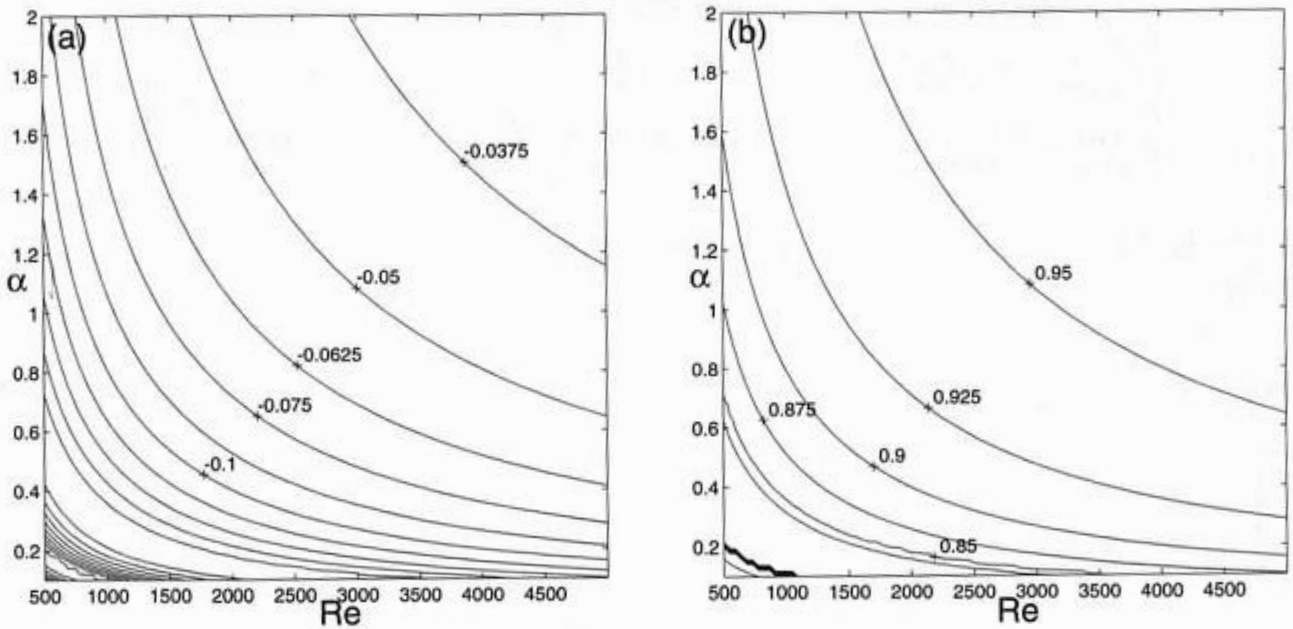
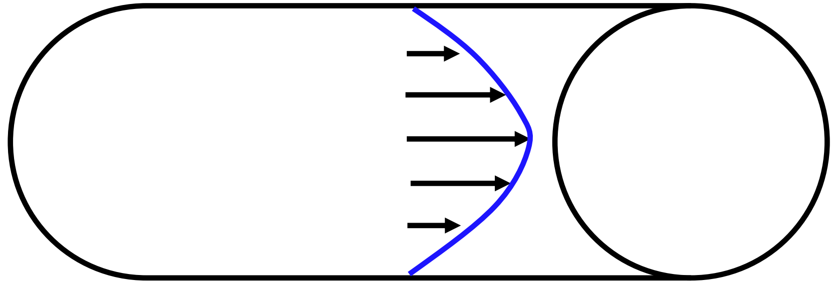
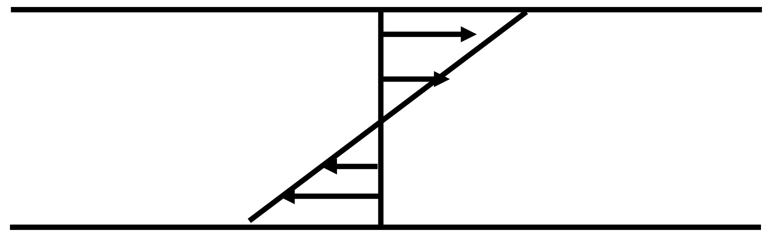


FIGURE 3.12. Contours of constant growth rate (a) and phase speed (b) for pipe Poiseuille flow.

Hagen Poiseuille

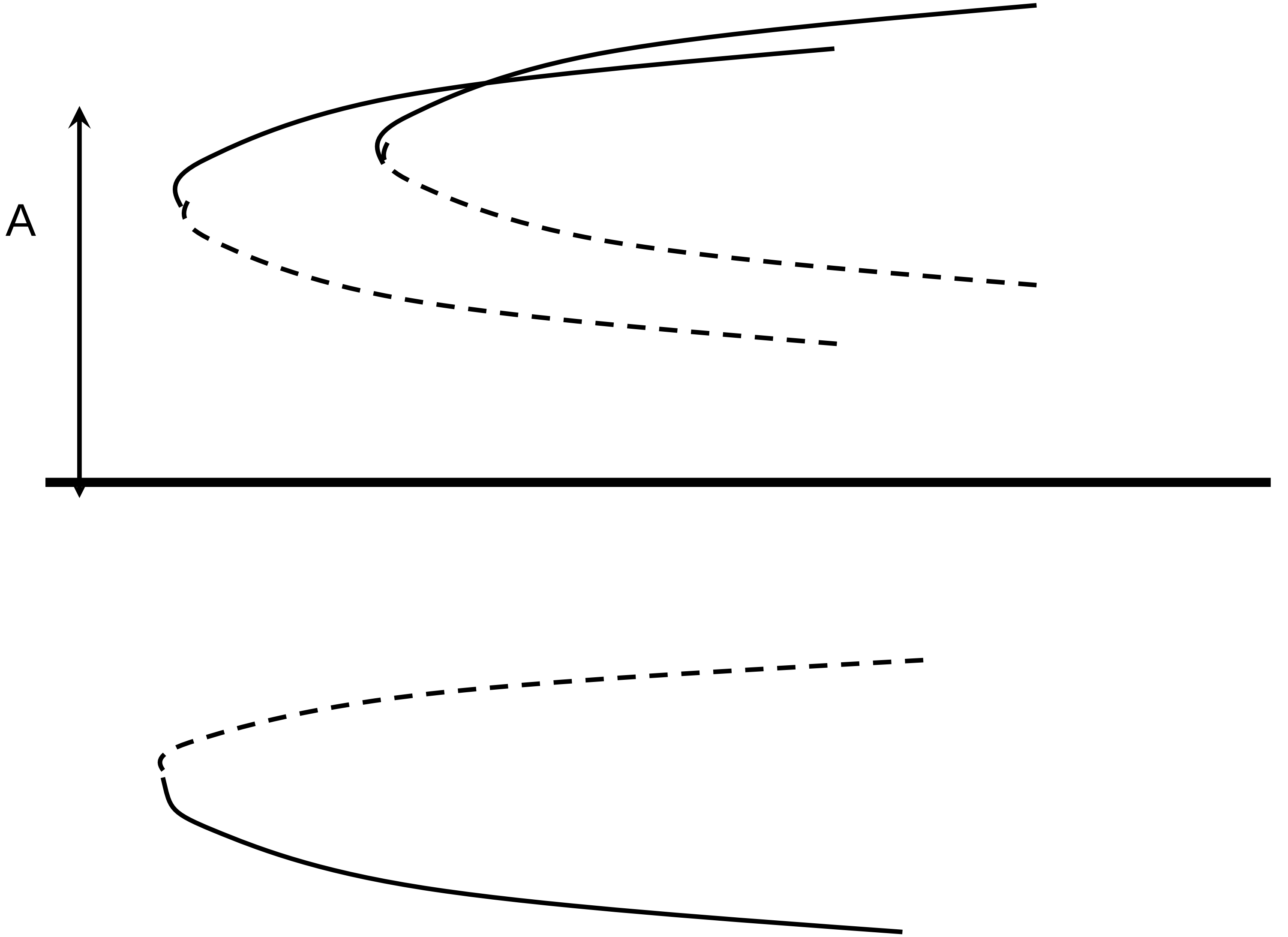


Couette



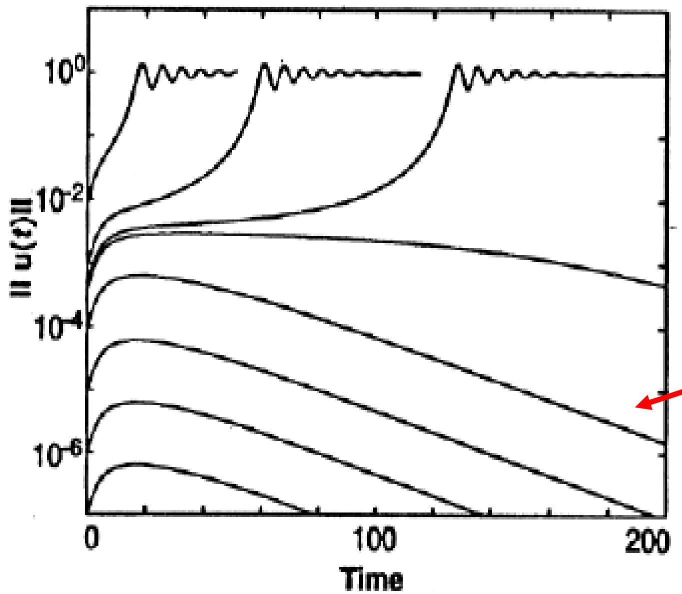
Flow type	Re_{exp}	Re_{lin}
Pipe flow	≈ 2000	∞
Plane Poiseuille flow	≈ 1000	5772
Plane Couette flow	≈ 360	∞

What about nonlinearities?



Edge states

Transient growth and bye pass transition

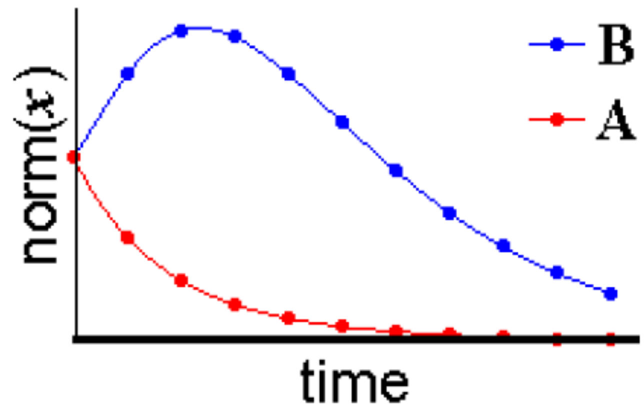


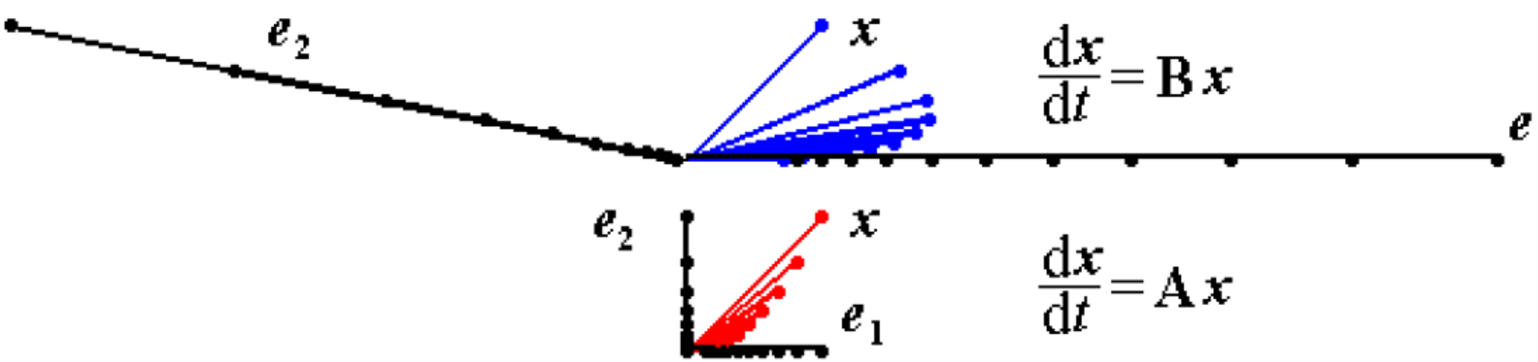
Eigenvalues only give you the asymptotic behavior

Conditional stability – subcritical bifurcation
Trefethen 1993

$$\mathbf{B} = \begin{bmatrix} -1 & 5 \\ 0 & -2 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$





$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \mathbf{B}\,\boldsymbol{x}$$

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \mathbf{A}\,\boldsymbol{x}$$

$$\frac{d\boldsymbol{u}}{dt} = B\boldsymbol{u} \qquad , \boldsymbol{u}(0) = \boldsymbol{u_0}$$

$$BB^{\dagger} \neq B^{\dagger}B$$

$$\Updownarrow$$

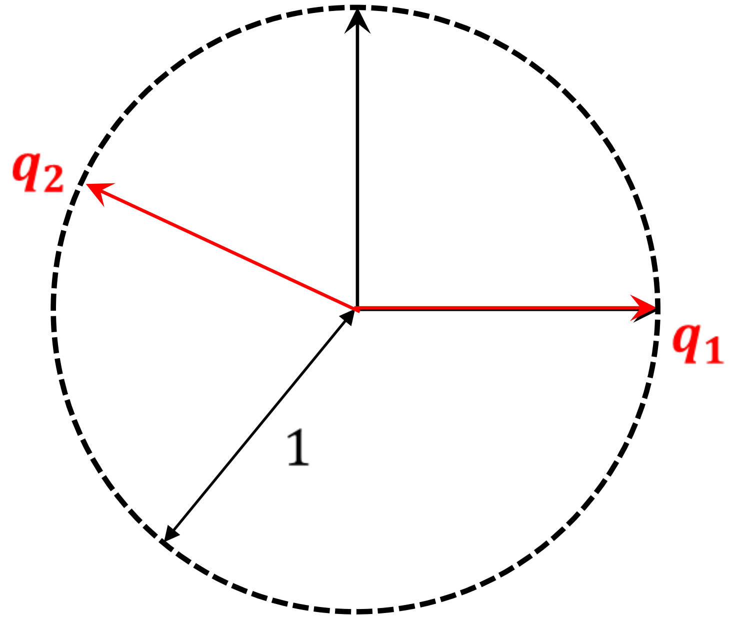
Eigenvectors non-orthogonals:

$$\frac{d\boldsymbol{u}}{dt} = B\boldsymbol{u} \qquad , \boldsymbol{u}(0) = \boldsymbol{u_0}$$

$BB^{\dagger} \neq B^{\dagger}B$

⇕

Eigenvectors non-orthogonals:

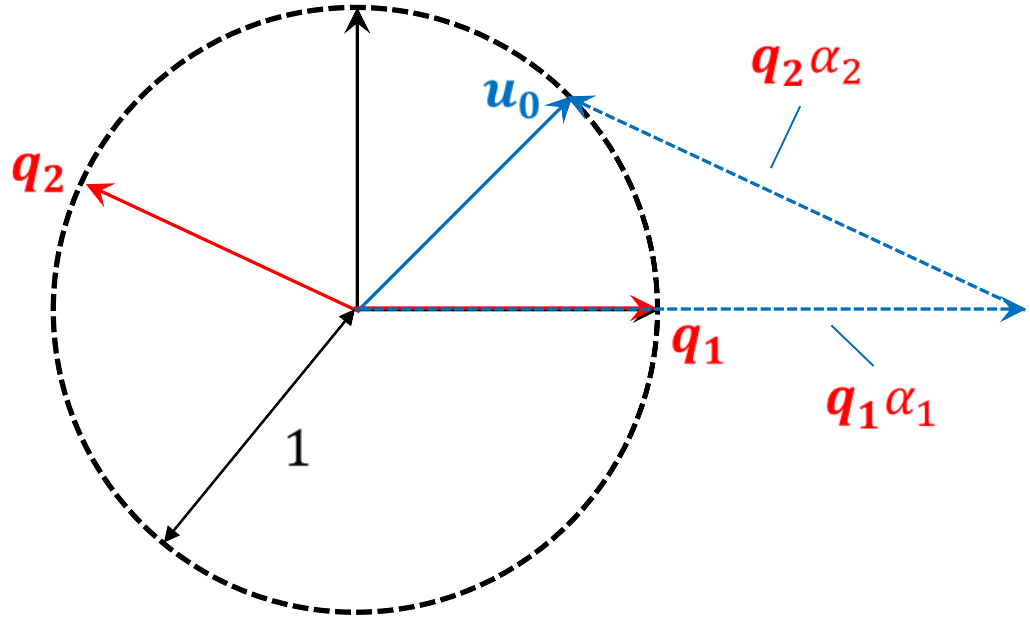


$$\frac{d\boldsymbol{u}}{dt} = B\boldsymbol{u} \qquad , \boldsymbol{u}(0) = \boldsymbol{u_0}$$

$BB^\dagger \neq B^\dagger B$

⇕

Eigenvectors non-orthogonals:



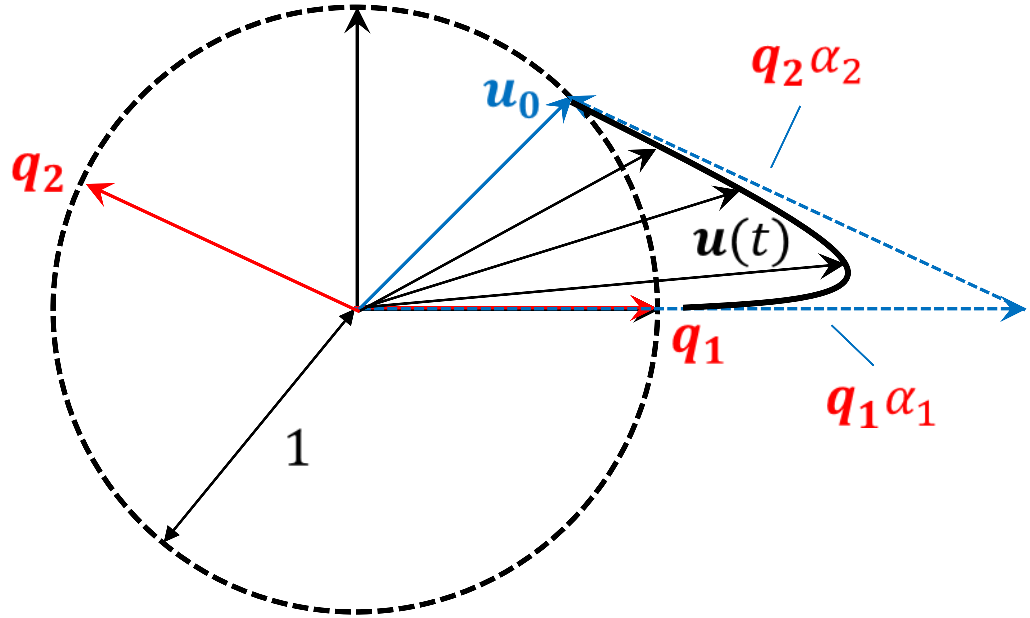
The projection of $\boldsymbol{u_0}$ on $\boldsymbol{q_1}$ is larger than $\boldsymbol{u_0}$!!

$$\frac{d\mathbf{u}}{dt} = B\mathbf{u} \qquad , \mathbf{u}(0) = \mathbf{u_0}$$

$BB^\dagger \neq B^\dagger B$

⇕

Eigenvectors non-orthogonals:



The projection of $\mathbf{u_0}$ on $\mathbf{q_1}$ is larger than $\mathbf{u_0}$!!

$$\mathbf{u}(t) = e^{\lambda_1 t}\mathbf{q_1}\alpha_1 + e^{\lambda_2 t}\mathbf{q_2}\alpha_2.$$

$\alpha_1 \gg 1$

Definition of optimal gain of a linear system

Linear system $$\frac{dq}{dt} = \mathbf{L}q$$

Initial condition $$q(t=0) = q_0$$

Optimal gain $$G(t) = \max_{q_0} \frac{||q||}{||q_0||} \quad ?$$

norm

Definition of optimal gain of a linear system

$$\frac{dq}{dt} = \mathbf{L}q$$

$$q(t=0) = q_0$$

Matrix exponential

$$q(t) = \exp(\mathbf{L}t)q_0$$

Definition of optimal gain of a linear system

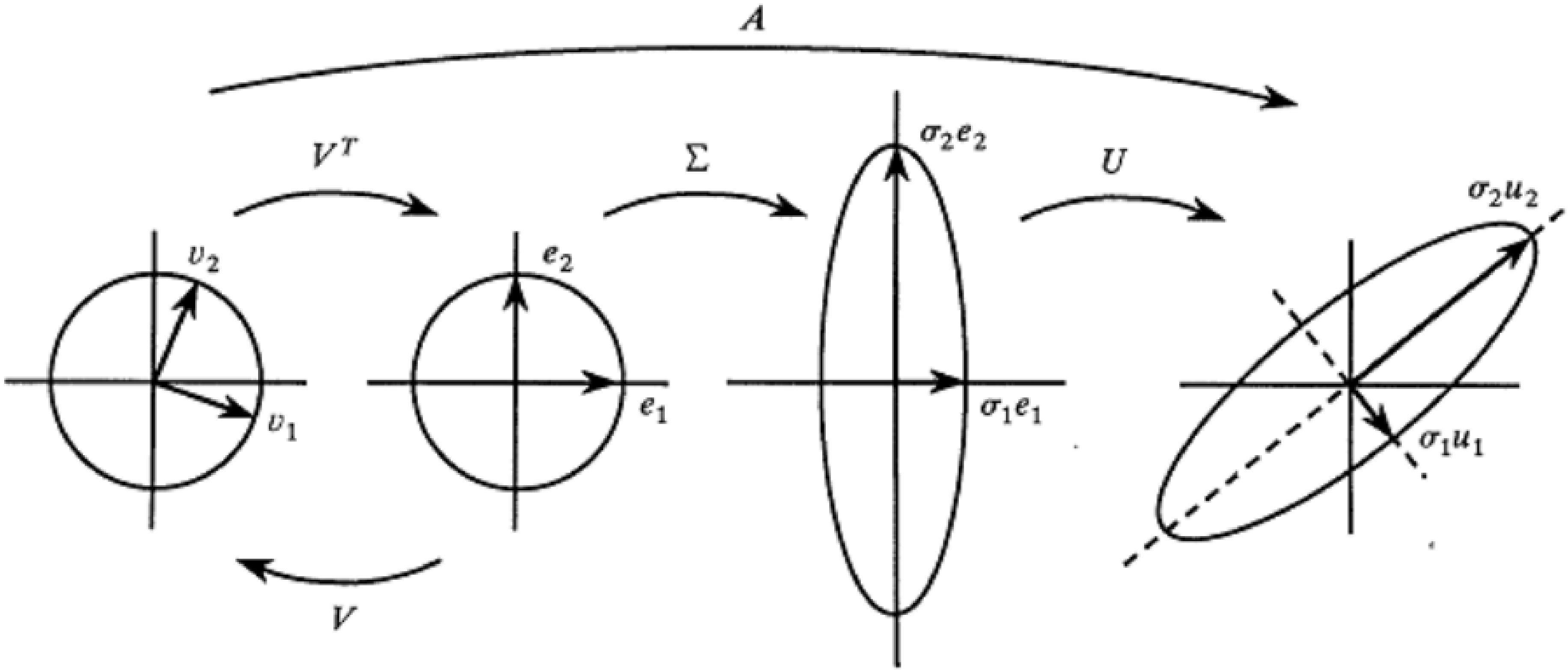
Optimal gain

$$G(t) = \max_{q_0} \frac{||q||}{||q_0||}$$

$$= \max_{q_0} \frac{||\exp(t\mathcal{L})q_0||}{||q_0||}$$

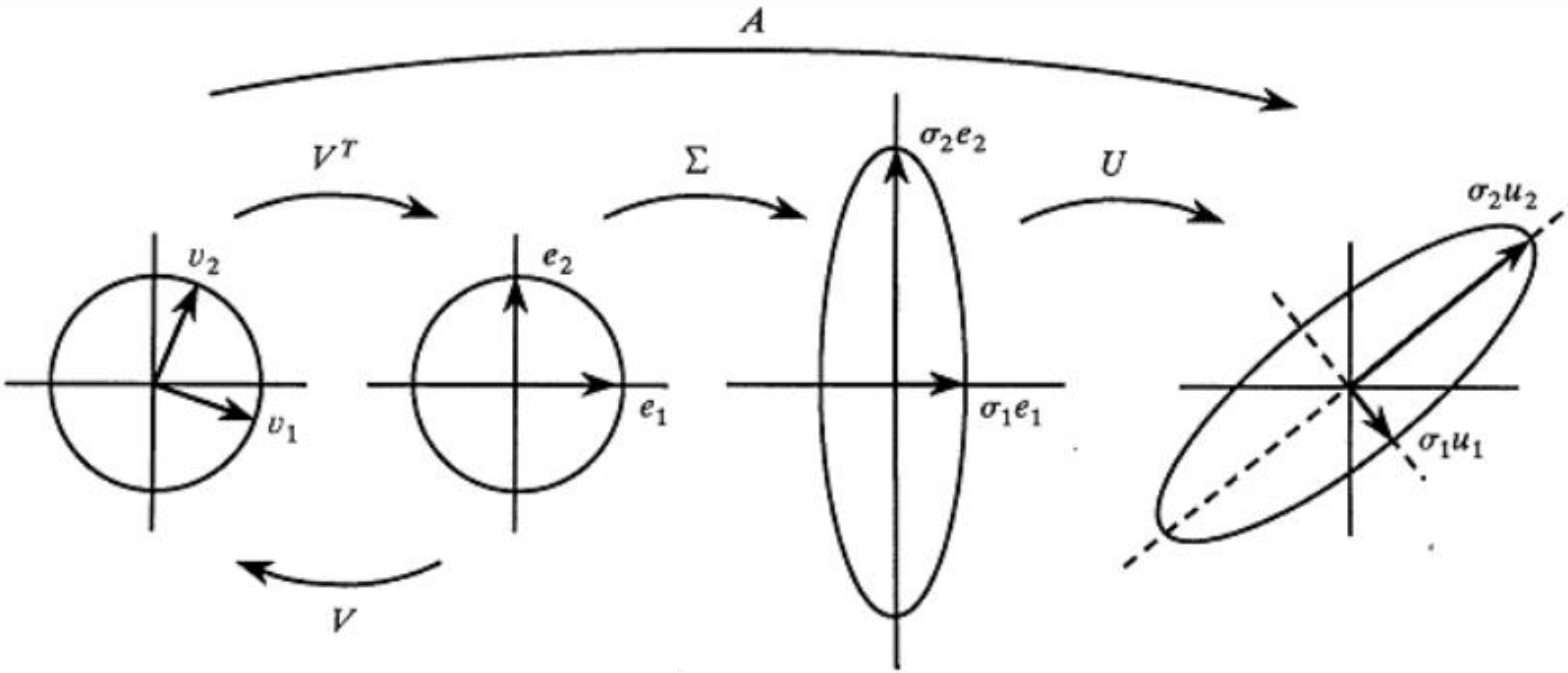
$$= ||\exp(t\mathcal{L})||$$

$$\|v_i\| = \|u_i\| = 1$$



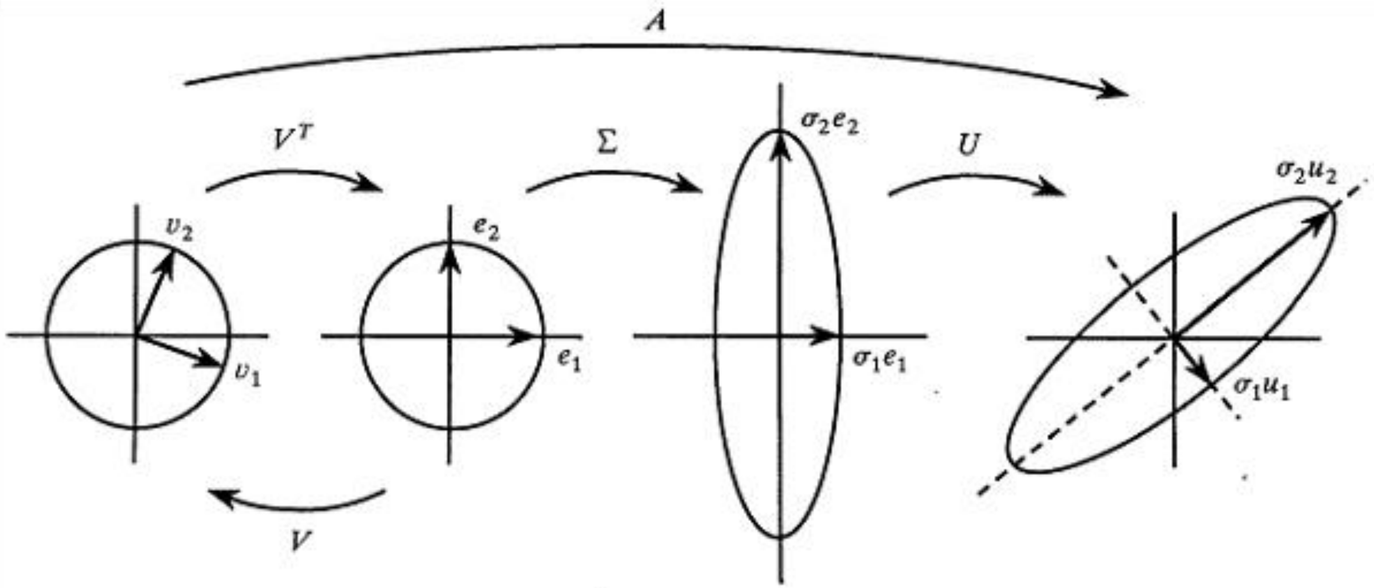
$$\sigma_2 = \|A\| = \max \|Ax\|, \ \ \|x\| = 1$$

$$\|v_i\| = \|u_i\| = 1$$



$$\sigma_2 = \|A\| = \max \|Ax\|, \; \|x\| = 1$$

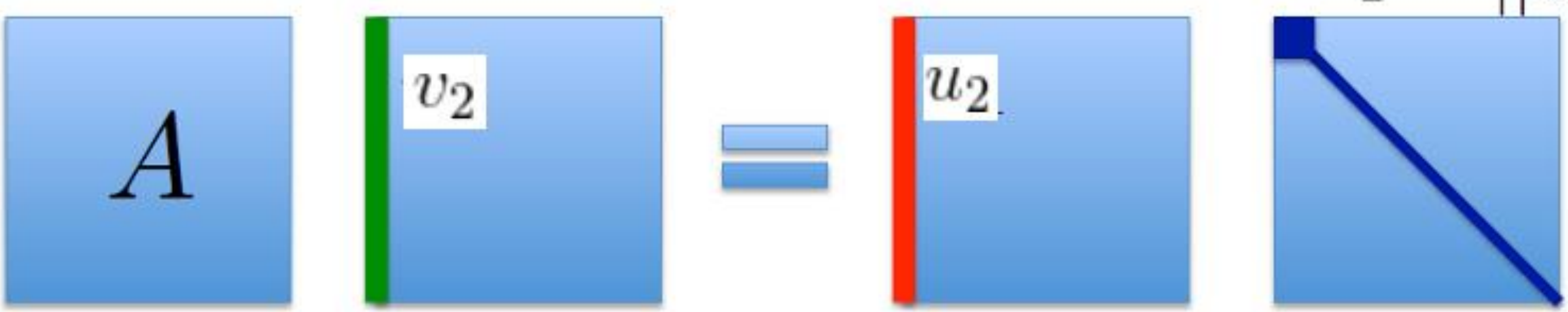
Associated singular pair : $Av_2 = \sigma_2 u_2$



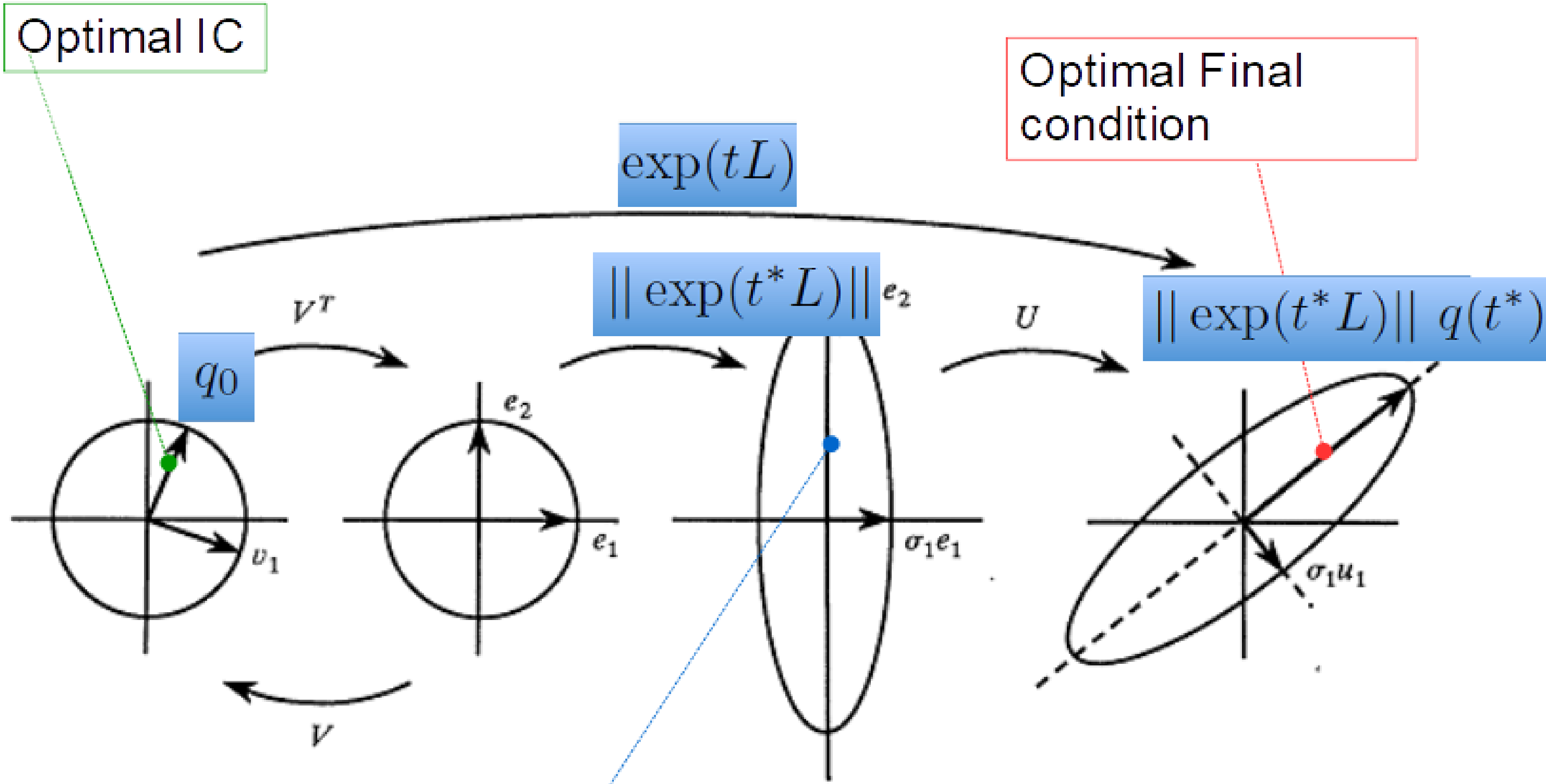
Singular value decomposition of a matrix

$$A = U\Sigma V^H \qquad AV = U\Sigma$$

$$\sigma_2 = ||A||$$



$$\exp(t^* L)\, q_0 = \|\exp(t^* L)\|\, q(t^*)$$



Optimal amplification = $G(t^*) = \|\exp(t^* L)\|$

Optimal initial condition

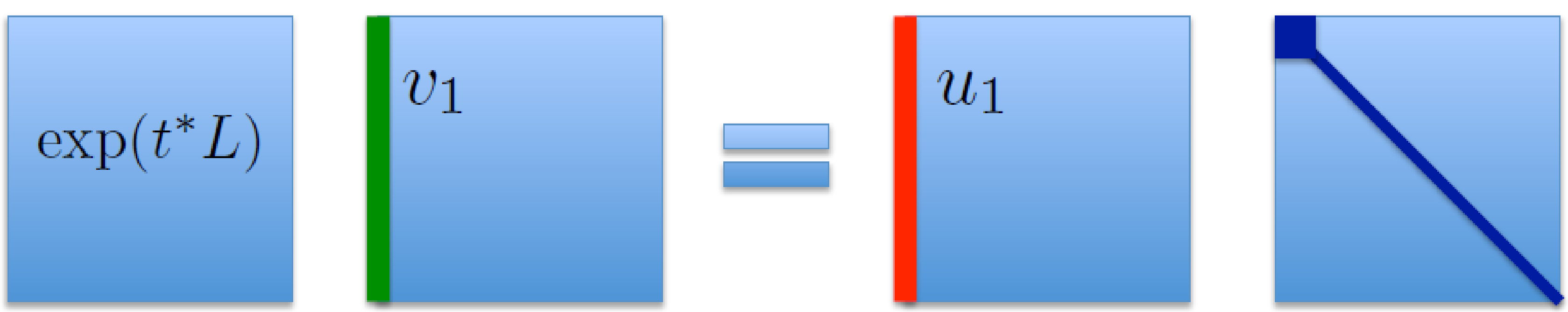
$$\exp(t^* L)\; q_0 = \;||\exp(t^* L)||\; q(t^*)$$

propagator input amplification output

Singular value decomposition of a matrix

$$\mathrm{svd}(\exp(t^* L)) = U \Sigma V^H$$

$G(t^*) = ||\exp(t^* L)||$



Optimal initial condition
left principal
singular vector

Optimal final condition
right principal
singular vector

A vous de jouer !

Documentation Examples Functions Apps Videos Answers

svd

Singular value decomposition

Syntax

```
S = svd(A)
[U,S,V] = svd(A)
[ __ ] = svd(A,"econ")
[ __ ] = svd(A,0)
[ __ ] = svd( __ ,outputForm)
```

Description

`S = svd(A)` returns the singular values of matrix `A` in descending order.

`[U,S,V] = svd(A)` performs a singular value decomposition of matrix `A`, such that `A = U*S*V'`.

The svd only computes the L^2 norm …

But sometimes the L^2 norm of the state vector $\boldsymbol{q}$ has no physical interpretation!

if $\boldsymbol{q} = \begin{pmatrix} v \\ \eta \end{pmatrix}$ the L^2 norm $\|\boldsymbol{q}\|^2{}_2 = \int_{\boldsymbol{\Omega}} |v|^2 + |\eta|^2 \, d\Omega$ means nothing !

if $\boldsymbol{q} = \begin{pmatrix} u \\ v \\ w \\ p \end{pmatrix}$ the L^2 norm $\|\boldsymbol{q}\|^2{}_2 = \int_{\boldsymbol{\Omega}} |u|^2 + |v|^2 + |w|^2 + |p|^2 \, d\Omega$

means nothing !

if $\boldsymbol{q} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$ the L^2 norm $\|\boldsymbol{q}\|^2{}_2 = \int_{\boldsymbol{\Omega}} |u|^2 + |v|^2 + |w|^2 \, d\Omega$

is the kinetic energy !

We need to find the proper norm, which, when measuring $\boldsymbol{q}$, gives something physical !

Optimal gain is associated to a norm

Kinetic energy written in v,η form

iαu=Dv-iβw and η=iβu-iαw

$$E(t) = \frac{1}{2k^2}\int_{\Omega}\left[|\mathcal{D}v|^2 + k^2|v|^2 + |\eta|^2\right]\, d\Omega$$

$$= \frac{1}{2}\int_{\Omega} |u|^2 + |v|^2 + |w|^2 \, d\Omega$$

Optimal gain is associated to a norm

Kinetic energy written in v,η form

iαu=Dv-iβw and η=iβu-iαw

$$E(t) = \frac{1}{2k^2}\int_\Omega \left[|\mathcal{D}v|^2 + k^2|v|^2 + |\eta|^2\right] \, d\Omega$$

$$= \|q\|_E^2 = \frac{1}{2k^2}\int_\Omega \begin{pmatrix} v \\ \eta \end{pmatrix}^H \begin{pmatrix} -\mathcal{D}^2 + k^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v \\ \eta \end{pmatrix} \, d\Omega$$

$$= \frac{1}{2k^2}\int_\Omega q^H M q \, d\Omega$$

energy matrix

Definition of optimal gain of a linear system

Cholevski dec. $M = F^H F$

$$||q||_E^2 = \frac{1}{2k^2}\int_\Omega q^H F^H F q \, d\Omega = \frac{1}{2k^2}\int_\Omega (Fq)^H F q \, d\Omega$$

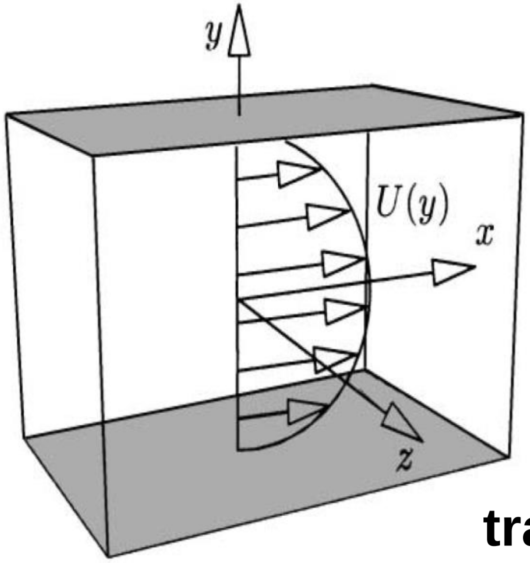
Optimal gain

$$G(t) = \max_{q_0} \frac{||q||_E}{||q_0||_E} = \max_{q_0} \frac{||Fq||_2}{||Fq_0||_2} = \max_{q_0} \frac{||F \exp(tL) q_0||_2}{||Fq_0||_2}$$

$$= \max_{q_0} \frac{||F \exp(tL) F^{-1} Fq_0||_2}{||Fq_0||_2} = ||F \exp(tL) F^{-1}||_2$$

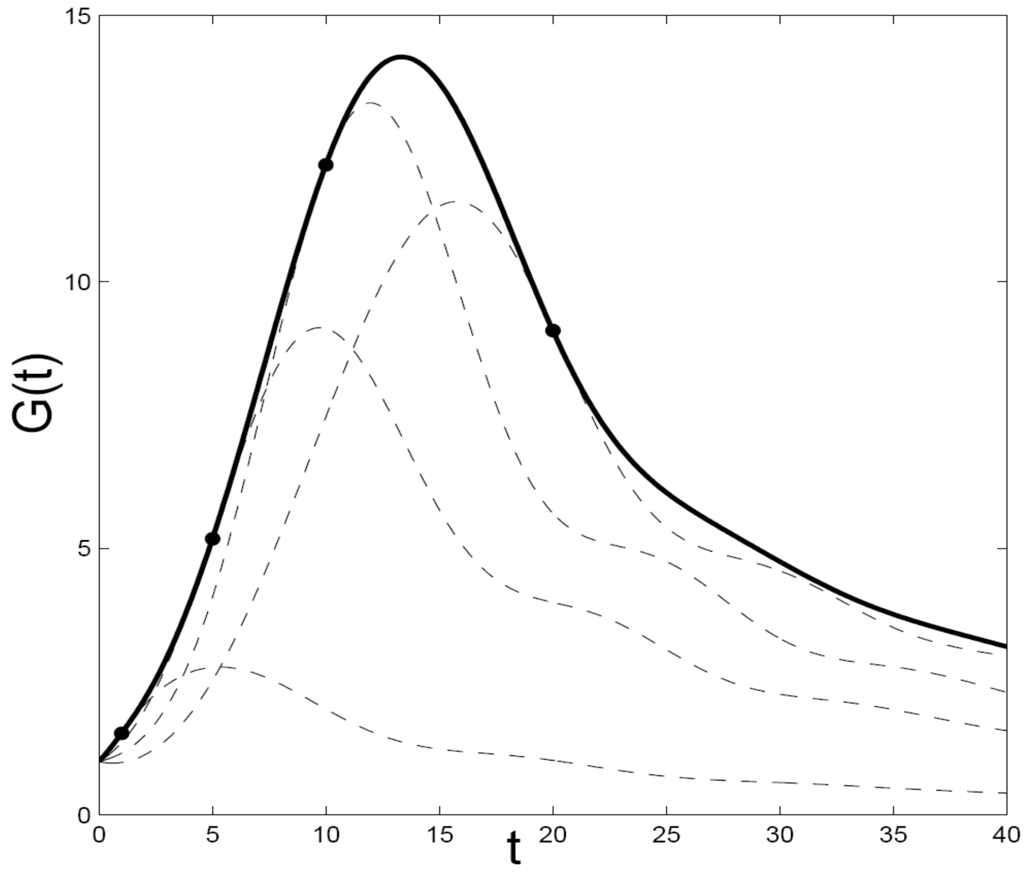
$= q'_0$

Plane Poiseuille flow



Linearly stable until Re=5772, but the transition is observed experimentally close to Re=1000-2000!

Orr mechanism (2D)

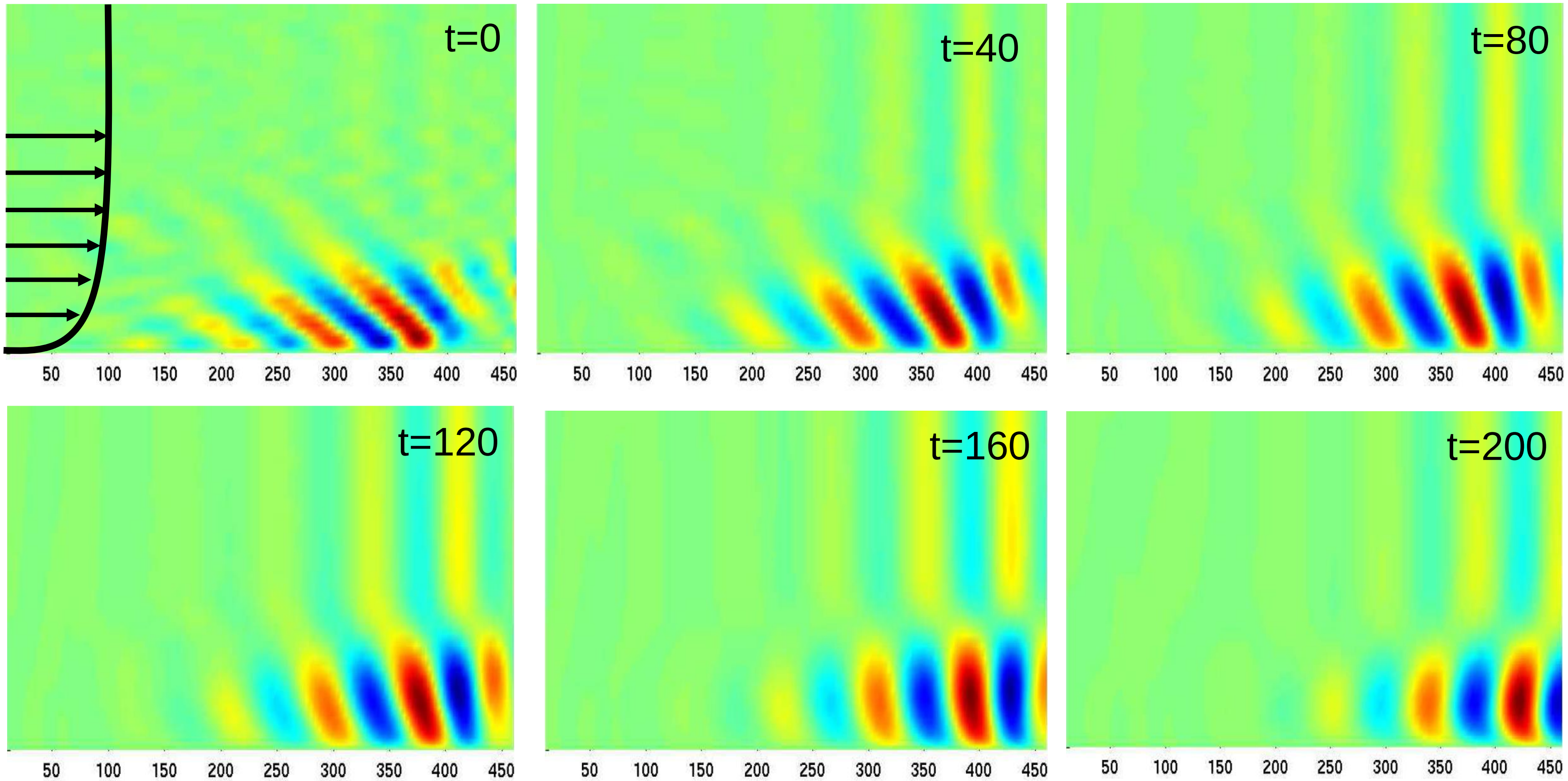


Amplification $G(t)$ for Poiseuille flow with $\mathrm{Re} = 1000, \alpha = 1$ (solid line)

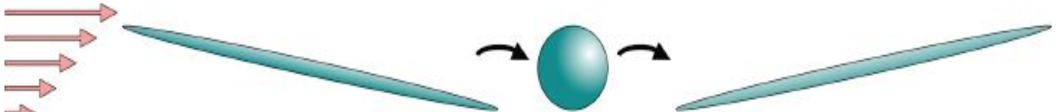
« By-pass transition »

Trefethen et al (1993), Buttler & Farrell (1993) , Schmid & Henningson (2001)

Optimal perturbation (here in boundary layer)

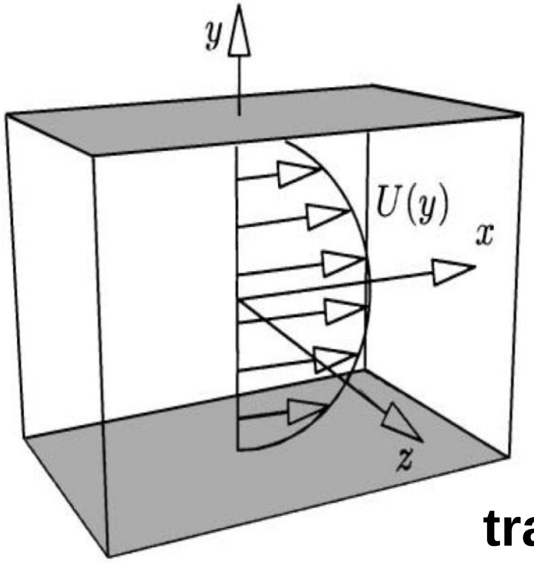


Orr mechanism



Plane Poiseuille flow

normal

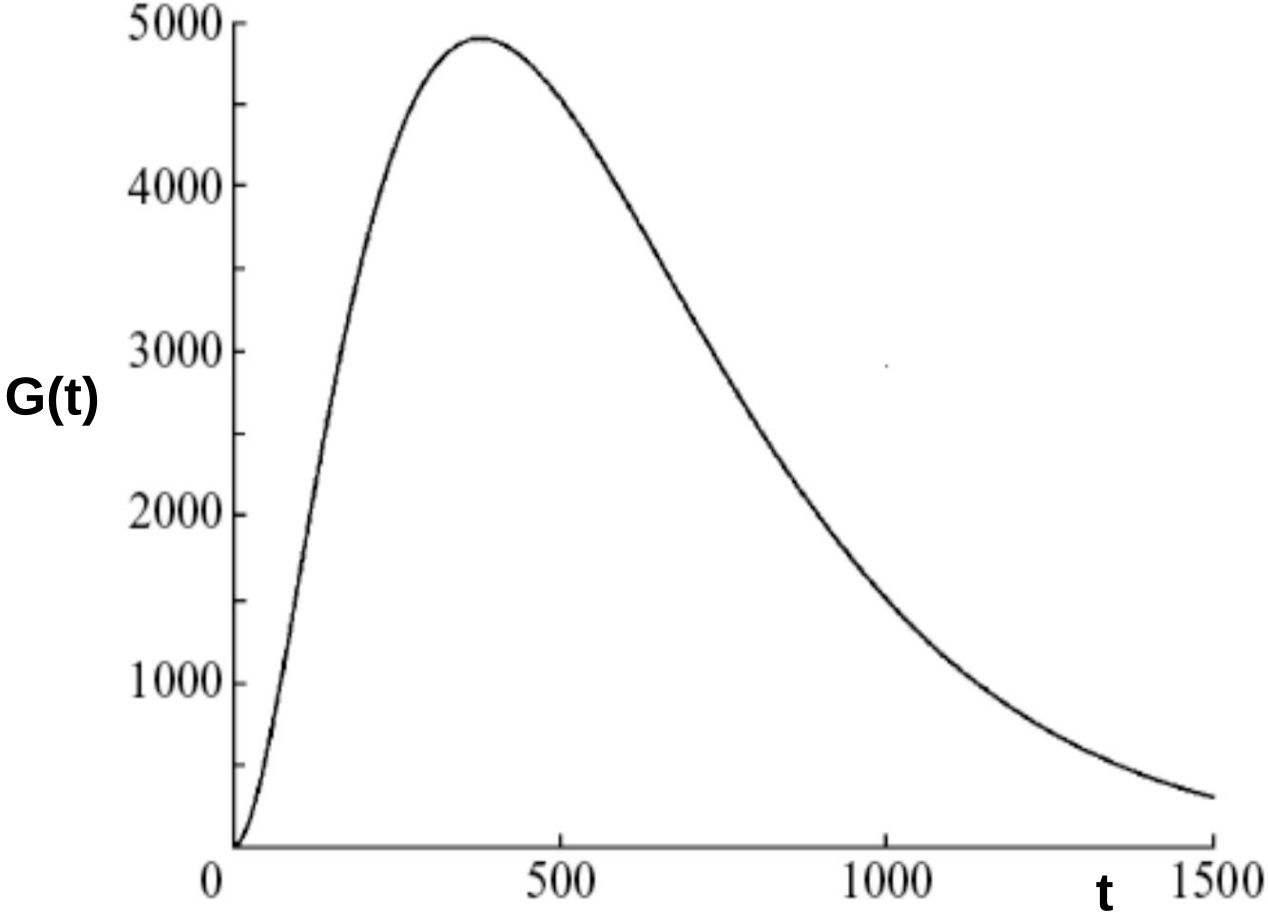


longitudinal, $e^{i\alpha x}$

transverse, $e^{i\beta z}$

Linearly stable until Re=5772, but the transition is observed experimentally close to Re=1000-2000!

Plane Poiseuille flow
Lift-up mechanism (3D)



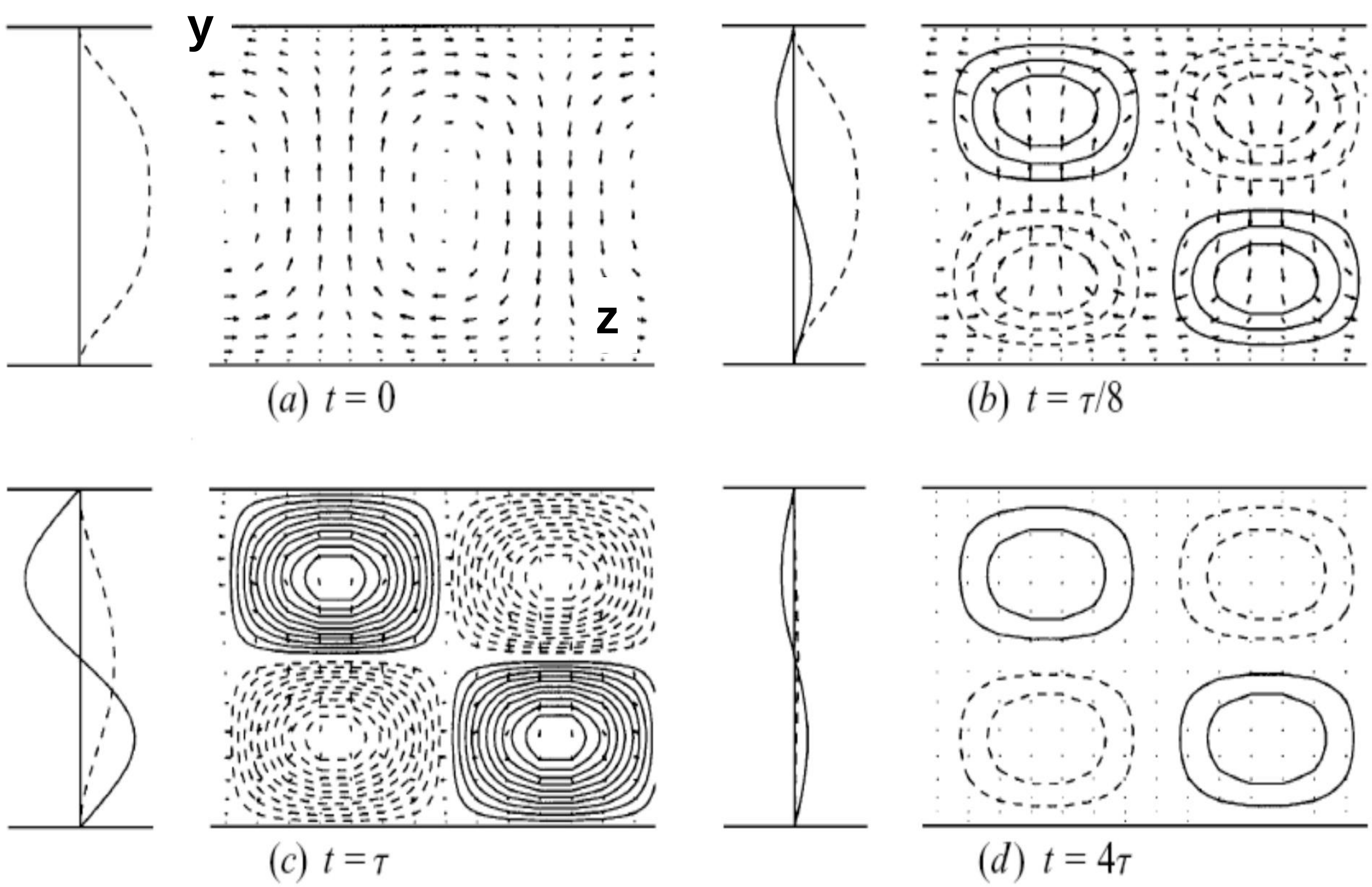
Re=5000, α=0, β=1

« By-pass transition »

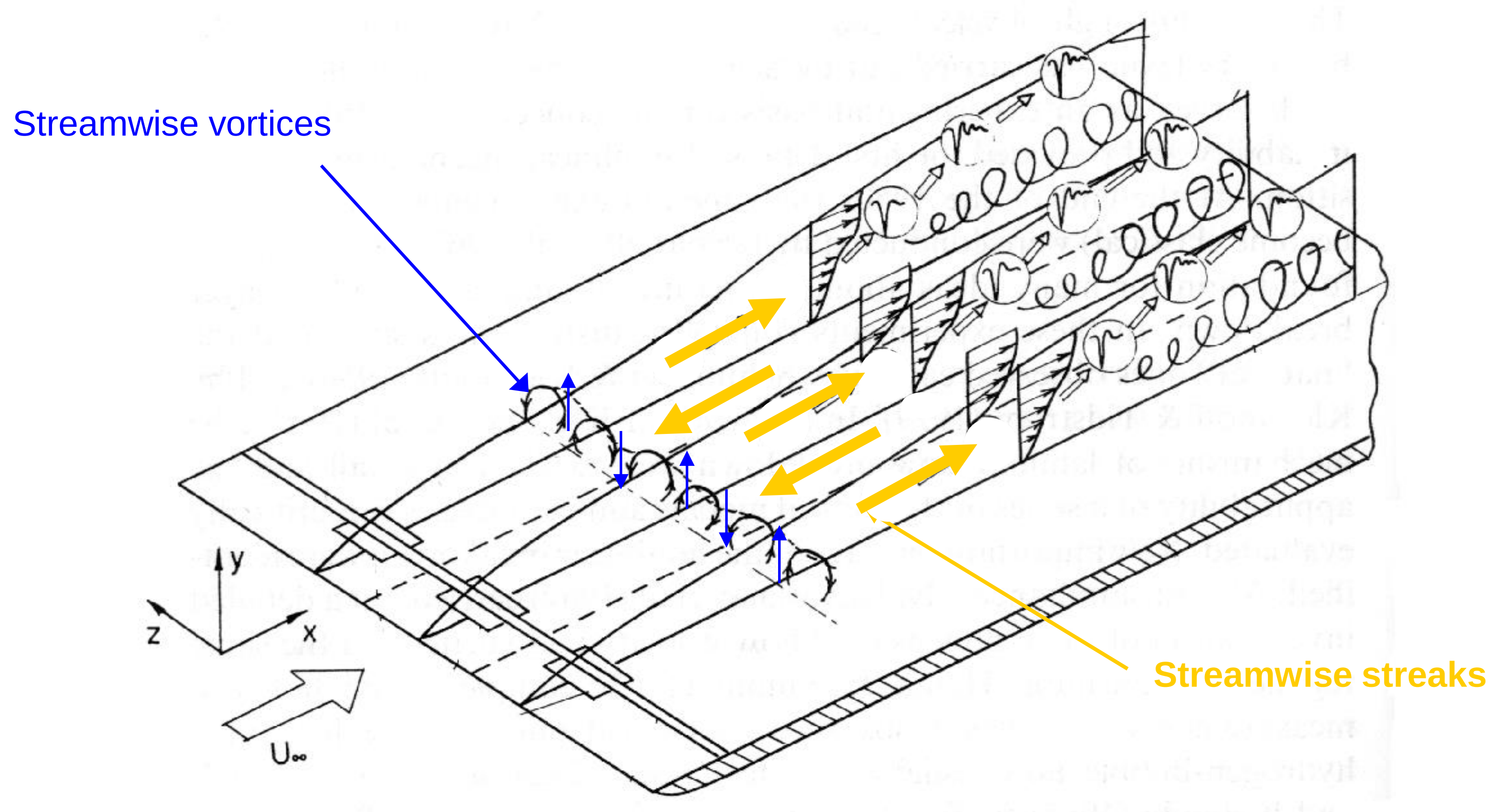
Trefethen et al (1993), Buttler & Farrell (1993) , Schmid & Henningson (2001)

Lift-up mechanisms

Optimal transformation of vortices into streaks



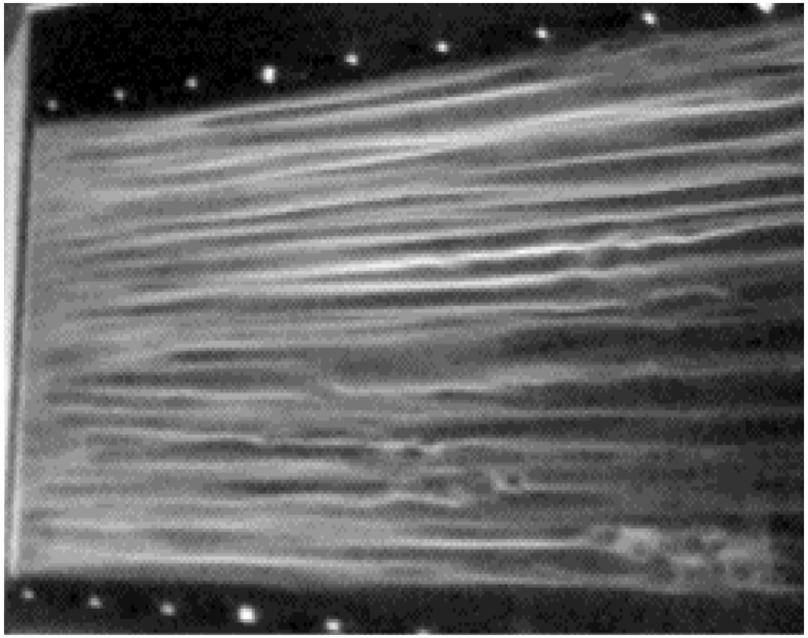
Lift-up mechanism



Schmid & Henningson.(2001)

Lift-up mechanism

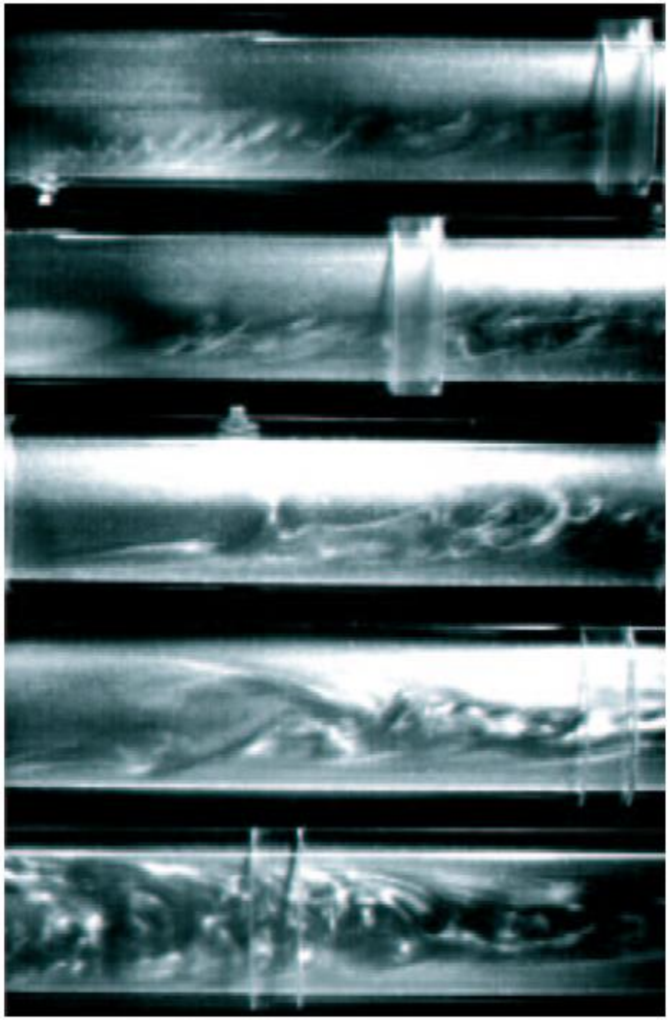
Optimal transformation of vortices into streaks



Alfredson & Matsubara (1996), streaky structures in the boudary layer

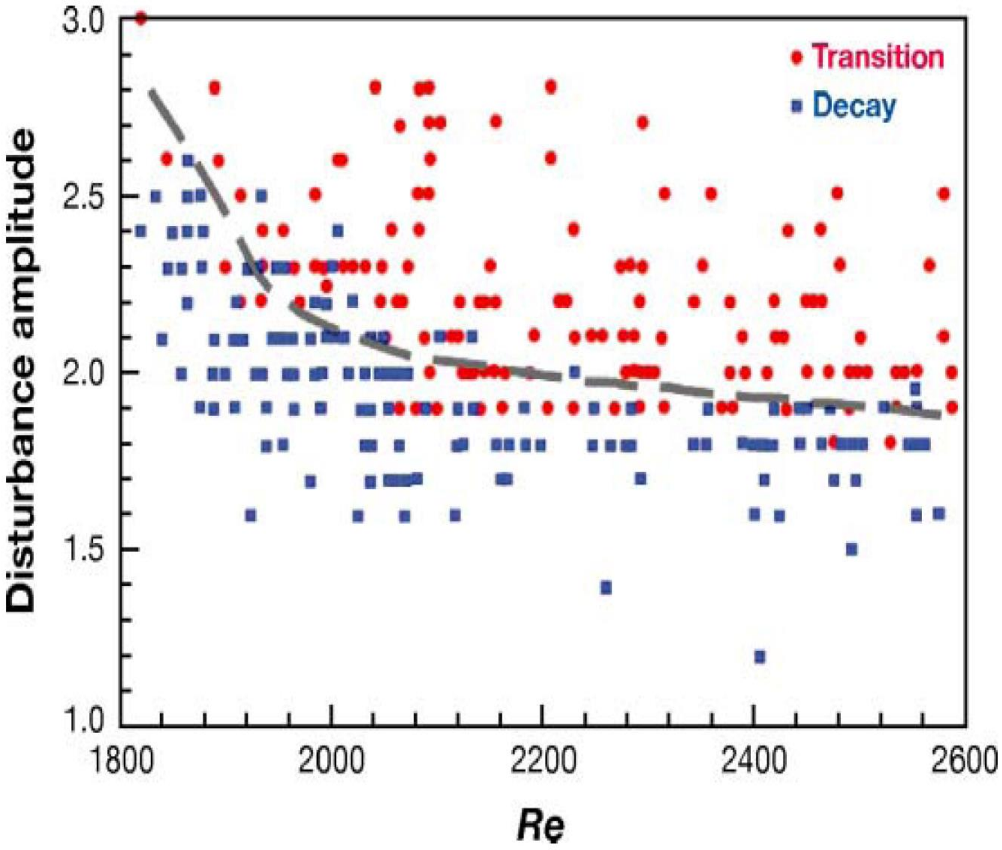
	$G_{\max}$ (10^{-3})	t_{max}	α	β
Plane Poiseuille	0.20 Re^2	0.076 Re	0	2.04
Couette	1.18 Re^2	0.117 Re	35/Re	1.6
Pipe	0.07 Re^2	0.048 Re	0	1
Boundary layer	1.50 Re^2	0.778 Re	0	0.65

Transition in Cylindrical Pipe Poiseuille flow



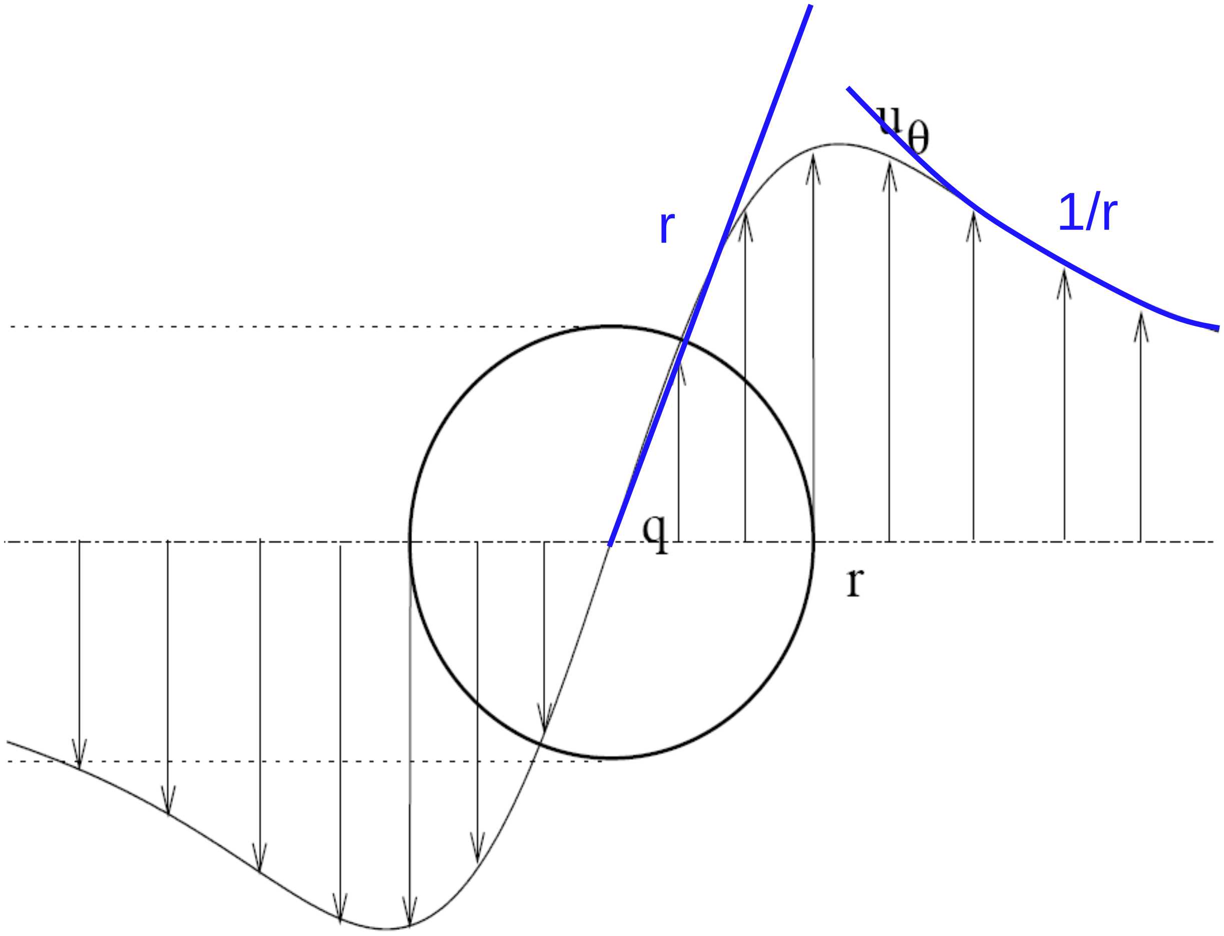
Mullin (2008)

Transition in Cylindrical Pipe Poiseuille flow



Mullin (2008)

Tourbillon de Lamb-Oseen



Stable lineairement!

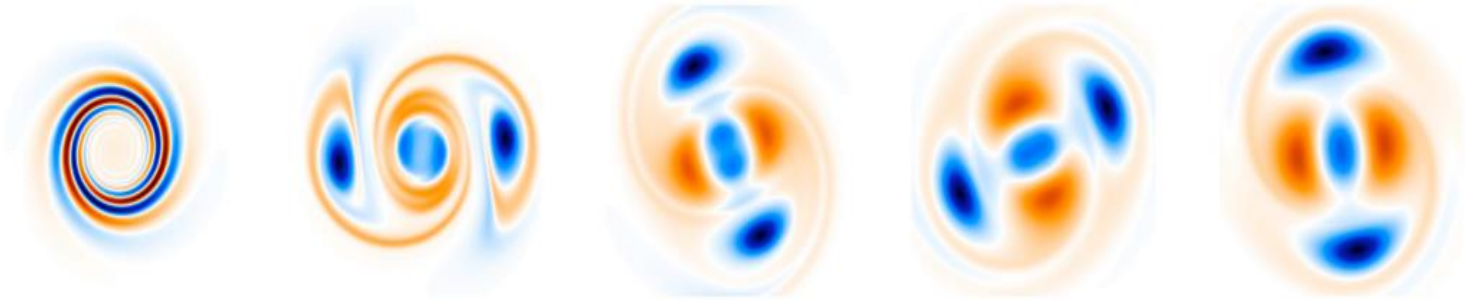
TOWARDS AN ELLIPTICAL STATE

Nonlinear evolution of the optimal perturbation with initial amplitude below a given threshold...

Re = 1000

TOWARDS AN ELLIPTICAL STATE

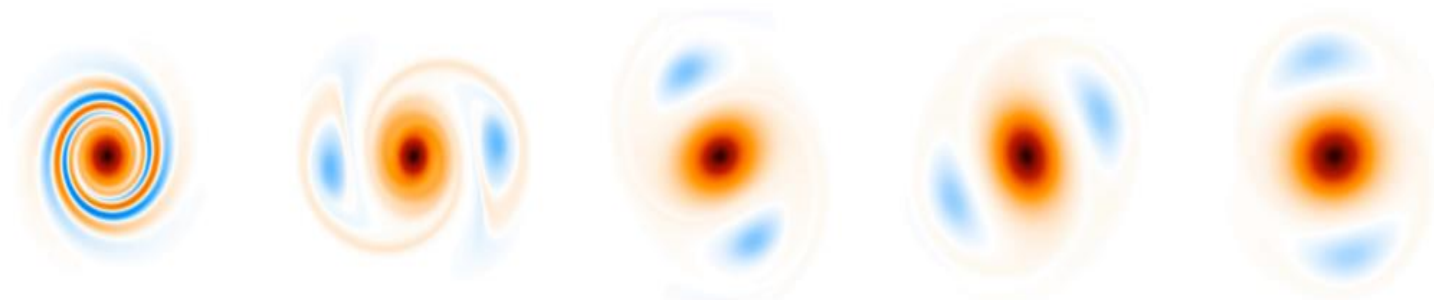
… and now ***ABOVE*** *the threshold*



Re = 1000

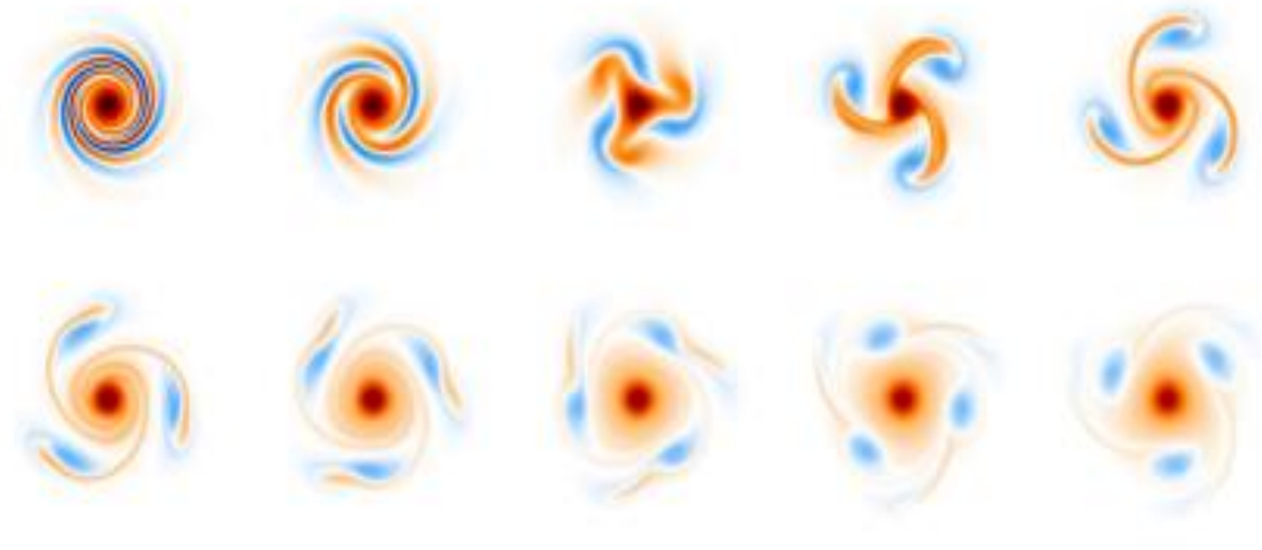
TOWARDS AN ELLIPTICAL STATE

Reconstructed flow

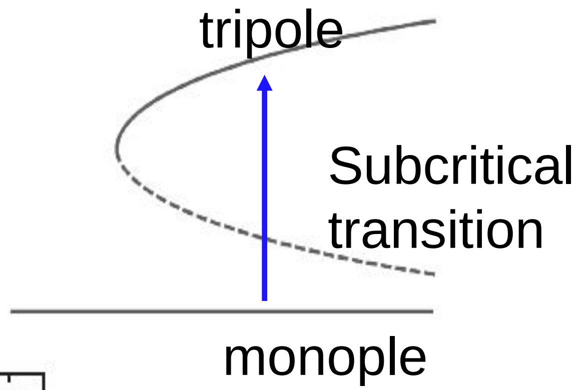


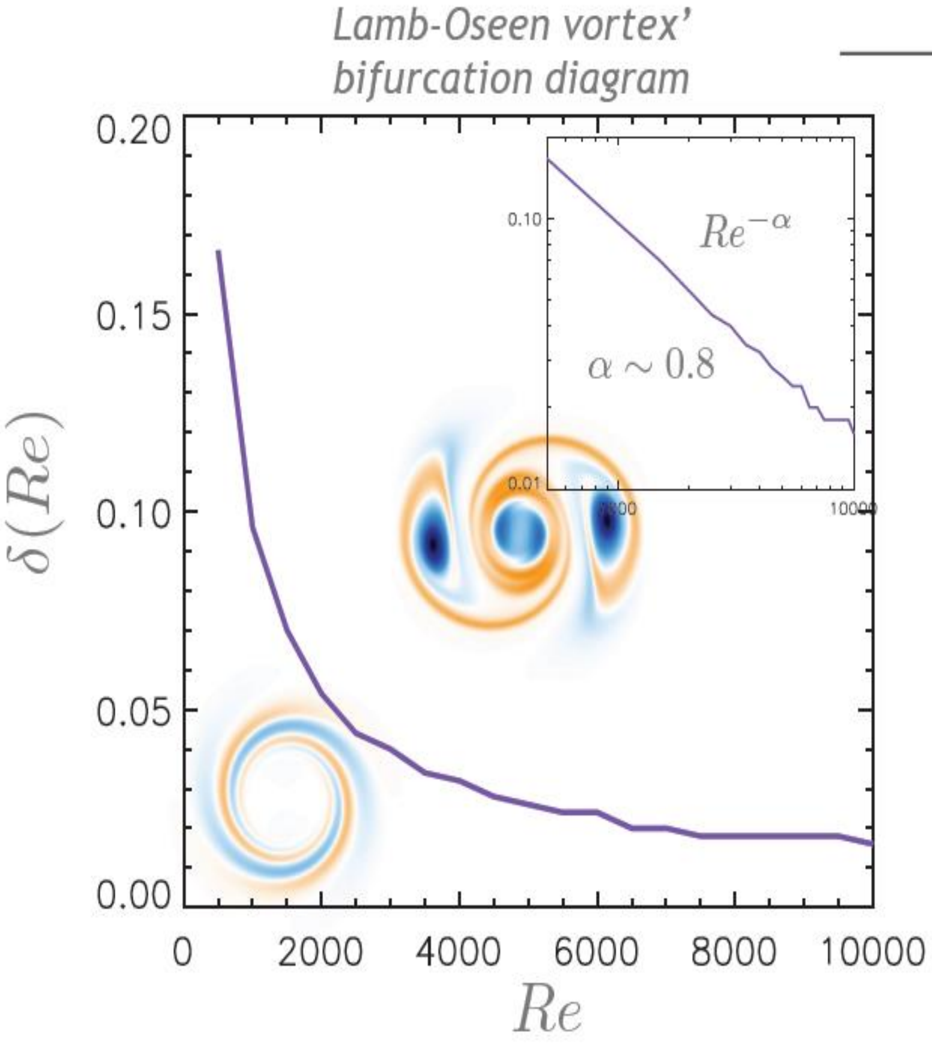
Re = 1000

Tripolar vortices?



BASIN OF ATTRACTION'S SHRINKAGE





Going further :

(on moodle)

Check for updates

Analysis of Fluid Systems: Stability, Receptivity, Sensitivity

Lecture notes from the FLOW-NORDITA Summer School on Advanced Instability Methods for Complex Flows, Stockholm, Sweden, 2013

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This article presents techniques for the analysis of fluid systems. It adopts an optimization-based point of view, formulating common concepts such as stability and receptivity in terms of a cost functional to be optimized subject to constraints given by the governing equations. This approach differs significantly from eigenvalue-based methods that cover the time-asymptotic limit for stability problems or the resonant limit for receptivity problems. Formal substitution of the solution operator for linear time-invariant systems results in the matrix exponential norm and the resolvent norm as measures to assess the optimal response to initial conditions or external harmonic forcing. The optimization-based approach can be extended by introducing adjoint variables that enforce governing equations and constraints. This step allows the analysis of far more general fluid systems, such as time-varying and nonlinear flows, and the investigation of wavemaker regions, structural sensitivities, and passive control strategies. [DOI: 10.1115/1.4026375]

1 Introduction and Motivation

Fluid systems are often described and characterized by their stability or receptivity behavior. Perturbations of infinitesimal amplitude that grow when superimposed on an equilibrium state of the flow render the base flow unstable; similarly, a flow that responds strongly when harmonically forced by an external excitation is referred to as receptive to this particular driving. Standard mathematical techniques have been devised to describe these fundamental questions of fluid dynamics: eigenvalue analysis for stability problems, and the resonance concept for receptivity problems. If the linearized equations exhibit at least one eigenvalue in the unstable half-plane, an instability is deduced; if the forcing frequency coincides with one of the eigenvalues of the linearized equations, a resonance is present in the flow.

Even though these techniques are valuable quantitative tools for the description of fluid problems, they have been found inadequate to account for the *full* behavior of many fluid systems. A property of the underlying equations, known as non-normality, allows for a far richer linear behavior than what can be measured by eigenvalues or resonances alone. By recasting the questions of

tutorial are available from the journal website and cover the majority of the concepts (and figures) treated in this article.

2 The Governing Equations

Even though the tools and techniques in this article readily apply to more complex flows, for sake of clarity we will consider the flow of an incompressible fluid confined by two walls. Two cases will be be treated: (i) the pressure-driven flow between two resting plates yielding a parabolic base-flow velocity profile (i.e., plane Poiseuille flow), and (ii) the flow induced by the two plates moving in-plane in opposite directions by the same speed producing a linear base-flow velocity profile (i.e., plane Couette flow). In either case, the base flow is given by the streamwise velocity component $U(y)$, that only varies in the normal (plate-to-plate) direction y. A sketch of the two flow cases, together with the coordinate system, is given in Fig. 1.

Linearizing the incompressible Navier–Stokes equations about the base flow $U(y)$ yields the following system of equations: